A Note on the Canonical Structure Theorem for Linear Systems

by

L. Weiss[1] and P. L. Falb[2]
Center for Dynamical Systems
Brown University
Providence, Rhode Island

1. Introduction.

The purpose of this note is to restate a little more accurately one part of the canonical structure theorem of Kalman [1,2] for linear differential systems. The restatement changes slightly the pattern of zeros in the coefficient matrix of the state vector. The originally stated form was perpetuated in the extension of the theorem by Weiss [3,4] and has found its way into the textbook literature [5].

For simplicity, only the time-invariant case is considered. A rigorous proof of the theorem for time-varying systems is contained in [6], based on some important facts established in [7] concerning matrices of constant rank whose elements are functions of time.

2. Results.

Consider a linear system

$$\text{(1)}\qquad \begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned}$$

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(One can avoid the appellatives "causal" and "anticausal" by adopting additional terminology such as "reachability" and "constructibility" in addition to "controllability" and "observability" [8]).

<u>Theorem:</u> (i) At each point t in time, there is a direct sum decomposition of the state vector $x(t)$ of (1) into four parts

$$x(t) = x_1(t) \oplus x_2(t) \oplus x_3(t) \oplus x_4(t)$$

in which $x_1(t)$ is controllable and unobservable, $x_2(t)$ is controllable and observable, $x_3(t)$ is unctrollable and unobservable, and $x^4(t)$ is uncontrollable and observable:

(ii) Corresponding to this decomposition is a coordinate transformation on the state space of (1) with respect to which the system matrices F,G,H have the form

$$A = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ 0 & F_{22} & F_{23} & F_{24} \\ 0 & 0 & F_{33} & F_{34} \\ 0 & 0 & 0 & F_{44} \end{bmatrix} \quad ; \quad G = \begin{bmatrix} G_1 \\ G_2 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$H = [0 \quad H_2 \quad 0 \quad H_4].$$

Past statements of this theorem and its extensions [1-5] have claimed that $F_{23} = 0$. While this might be aesthetically satisfying in some sense, <u>it is not possible, in general, to transform an arbitrary</u> F <u>matrix to the above canonical form in which</u>

$F_{23} = 0$.

The proof of this statement is provided by the following counterexample. Let

(3) $$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad ; \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad H = [1 \quad 0].$$

The system (1) with F, G, H as in (3) is already in the canonical form (2). It has a subsystem

(4) $$\dot{x}_2(t) = x_2(t)$$

which is uncontrollable and unobservable; and it has another subsystem

(5) $$\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

which is controllable and observable. (In making the latter determination, the vector $x_2(t)$ in (5) is viewed as a forcing function and conventional definitions of observability involve considering the system with all forcing functions set to zero.)

It is perfectly evident, in any case, that no real coordinate transformation exists which can basically change the structure of (3).

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